An approximate equation for the spectrum of a conserved scalar quantity in a turbulent fluid

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The formula for transfer of $\overline{\theta}^2$-stuff across the θ-spectrum, which was obtained by Batchelor, Howells & Townsend (1959) for wave-numbers at which molecular conduction is important, is extended here to smaller wave-numbers by means of a simplified general picture of the mechanism involved. The interaction of velocity and temperature fields is represented by a combination of eddy conductivity due to the smaller eddies, and a straining action due to the larger eddies, and this leads to an approximate equation for the θ-spectrum, for a fluid of arbitrary Prandtl number, over at least the equilibrium range of wave-numbers.

An earlier paper (Batchelor *et al.* 1959) on the spectrum function of a conserved scalar quantity θ (such as temperature) in a turbulent fluid was concerned with the case of small Prandtl number, and gave a formula for an effective eddy conductivity due to low Péclet number components of the flow. This eddy conductivity κ_e due to velocity Fourier components having wave-numbers greater than n is such that the rate of transfer of $\overline{\theta}^2$-stuff to wave-numbers greater than n is $2\kappa_e$ times the mean square gradient of θ associated with wave-numbers less than n, and was found to be

$$\kappa_e(n) = \frac{2}{3}\int_n^\infty \frac{E(n)}{\kappa n^2}\,dn \quad \text{provided} \quad n \gg \epsilon^{\frac{1}{4}}\kappa^{-\frac{3}{4}}, \tag{1}$$

where κ is the molecular conductivity, ϵ and $E(n)$ are the rate of dissipation, and spectrum function, of kinetic energy.

We notice that the eddy conductivity is inversely proportional to the molecular conductivity—in other words the effectiveness of the Fourier components in convecting heat down a gradient, or in transferring $\overline{\theta}^2$-stuff from lower wave-numbers, is limited by molecular conduction. The reason for this can be seen from an adaptation of the usual 'mixing-length' argument. A blob of hot fluid of size l, convected into colder fluid, loses its excess heat by conduction, in a *time* $l^2\kappa^{-1}$, whereas in a high Péclet number flow, where conduction is not important, the excess heat is lost in a *distance* l. In the latter case the eddy conductivity due to wave-numbers greater than $n(\ll \epsilon^{\frac{1}{4}}\kappa^{-\frac{3}{4}})$ is given by a formula such as

$$\int_n^\infty \left\{\frac{E(n)}{n^3}\right\}^{\frac{1}{2}} dn. \tag{2}$$

Now it is of interest to obtain one formula which includes both expressions (1) and (2). To do this we start from (1), which is asymptotically exact, and consider how it is to be extended to Péclet numbers which are not small. Here the transfer of $\overline{\theta}^2$-stuff from lower wave-numbers by velocity Fourier components is limited not only by molecular conduction but by the interference of other Fourier components, and, to make use of (1), we represent this interference as the effect of an eddy conductivity due to larger wave-numbers. That is, we replace κ in the integrand of (1) by $\kappa + \kappa_e(n)$

$$\kappa_e(n) = \frac{2}{3}\int_n^\infty \frac{E(n)}{n^2\{\kappa+\kappa_e(n)\}}dn,$$

which leads to

$$\kappa_e(n) = \left[\kappa^2 + \frac{4}{3}\int_n^\infty \frac{E(n)}{n^2}dn\right]^{\frac{1}{2}} - \kappa. \tag{3}$$

At low Péclet numbers, equation (3) is equivalent to (1), and at high Péclet numbers it agrees sufficiently well with (2). However, in (2) the square-root comes inside the integral, because of the supposition that each small element of the range of wave-numbers from n to ∞ should make a separate and similar contribution to the eddy conductivity (carrying over by analogy the argument for eddy viscosity summarized in Batchelor 1953, §6.6). But this supposition implies that (2) should hold for all wave-numbers, in contradiction to (1), and when it is applied to the calculation of spectra the asymptotic forms obtained for $n \to \infty$ are negative powers of n, which are difficult to accept because they are inconsistent with the existence of all derivatives of the scalar field. In the argument leading to (3), the effect of each Fourier component involves the others, and this feature appears in the result, where the integral is inside the square root. The one general picture of the mechanism (although it is simplified) leads to a formula which includes the two limiting expressions, each in its range of validity.

The eddy conductivity mechanism is not adequate to describe the entire transfer of $\overline{\theta}^2$-stuff, since it neglects the straining action of the larger eddies, which is responsible for the transfer beyond the viscous cut-off in the case of large Prandtl number (Batchelor 1959). The additional quantity which must be introduced is a rate of strain due to wave-numbers less than n, say $\zeta(n)$. Then using the results of the above reference, and making the same sorts of approximation as are involved in the eddy conductivity arguments, we write the contribution from this mechanism to the rate of transfer of $\overline{\theta}^2$-stuff across wave-number n as $n\zeta(n)\Gamma(n)$, where $\Gamma(n)$ is the spectrum function of θ. $\zeta(n)$ should be proportional to

$$\left[\int_0^n n^2 E(n)\,dn\right]^{\frac{1}{2}},$$

and the constant of proportionality must be chosen to be about $1/\sqrt{2}$, in order to make $\zeta(\infty)$ agree with the value $\frac{1}{2}\sqrt{(\epsilon/\nu)}$ used in the paper quoted for the quantity denoted there by $-\gamma$.

Combining these effects, we can write an equation for $\Gamma(n)$ which should be valid in the same approximate way for all wave-numbers in the equilibrium range (and possibly for part of the energy-containing range) and for all Prandtl numbers:

$$\frac{\partial}{\partial t}\int_0^n \Gamma(n)\,dn+\left[\frac{1}{2}\int_0^n n^2E(n)\,dn\right]^{\frac{1}{2}} n\Gamma(n)+2\left[\kappa^2+\frac{4}{3}\int_n^\infty \frac{E(n)}{n^2}\,dn\right]^{\frac{1}{2}}\int_0^n n^2\Gamma(n)\,dn = 0. \tag{4}$$

It might be expected that something similar could be done for kinetic energy spectrum of turbulence—certainly the simple eddy viscosity approach is in error beyond the viscous cut-off. But there are two differences, one arising from the vector nature of the velocity field, and the other from the fact that it is interacting with itself. The problem of the magnetic field in a turbulent conducting fluid (when magnetic forces are not important) also differs from that of the scalar field in the first way, but not in the second; it is possible that its study may help elucidate some of the difficulties of the kinetic energy transfer in turbulence.

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